Open Proofs and Logic 2018 MAΘ National Convention Solutions

1. C	4. B	7. E (75)	10. A	13. A
2. B	5. D	8. C	11. E (44)	14. C
3. D	6. D	9. C	12. A	15. B

1. **C**: Drawing out a truth table:

x	y	xy’ + y
0	0	0
0	1	1
1	0	1
1	1	1

We can see that this function can be rewritten as x OR y.

2. **B:** The problem in statement two when we take the derivative is that we fail to consider that the number of x’s is also changing. This gives us an extra dx, which would make the left-hand side 2dx as well.

3. **D**: This is a direct application of the intermediate value theorem.

4. **B**: This is a direct application of the mean value theorem.

5. **D:** This Boolean function is simply an XOR of all the variables. XOR is 1 when there are an odd number of 1s input to it and 0 when there are an even number of 1s input to it.

6. **D**: Step 4 is the first to be incorrect because it fails to consider the +C of each integral when taking the difference.

7. **E(75)**: First, we can take every integer between 1 and 100 which is not a multiple of 3, which gives us 67 different values. Now since we have no multiples of 3, we can take all the multiples of 9 which are not multiple of 27 because any multiple of 9 is 3x a multiple of 3, which we do not have. But multiples of 27 are 3x multiples of 9, which we now have, giving us 74 values. Finally, we can take 81 since 27 is not in our set. This gives us a total of 75 values in our set.

8. **C**: There are four distinct possibilities for the order and gender of my children. Labeling Old first then Young: GG, GB, BG, BB. We are given the information that the younger one is a boy. This means that there are only two possibilities for my children: GB, BB. Therefore, the probability that I have two boys is ½.

9. **C**: Step 3 is incorrect because it assumes that complex exponentials are usable in the same way that real exponents are. This is unfortunately not the case, as clearly, $e^x \neq 1$ for all x.

10. **A**: First, note that 115 people of the 120 have already been included in one of the categories. Next, note that if everyone gave a valid permutation, if they guessed 4 of the digits correctly, there is only 1 leftover digit that they have not guessed which must also be correct. This means that 0 people guessed exactly 4 digits correctly and all 5 remaining people guessed all 5 digits of my permutation correctly.

11. **E(44)**: This question is simply asking for the number of derangements of a set of 5, which is 44.

12. **A**: Note that for any given arrangement, we can swap dogs for cats and get another arrangement. This means that the number of arrangements in which a dog places first is equal to the number of arrangements in which a cat places first. If the highest-ranking dog were not first, but second, then that would imply that a cat placed first. However, a cat placing first does not imply that the highest-ranking dog places second. This means that the set of races in which the highest-ranking dog places second is a subset of the set of races in which a cat places first. However, since we know that there are an equal number of races in which a cat places first and a dog places first, this means that there are more races in which a dog places first than the highest-ranking dog places second. The same logic can be used to eliminate all other possible answers.

13. **A**: Given our knowledge of question 12, the $10 should be wagered for the highest-ranking dog being first. We can again use similar logic to above to show that the lower the highest ranked dog is, the lower the probability that the scenario occurs. Thus, the ordering 1,2,3,4 for our wagers from low to high will give us the highest expected value.

14. **C**: For any given probability distribution function, the total area underneath it must equal one. The area underneath f(x) is $\int_1^\infty Cx^{-a}dx = C \cdot \left(\frac{1}{1-a}\right) \cdot [\infty^{1-a} - 1^{1-a}] = \frac{C}{a-1} = 1$. Thus, $C = a - 1$. Note that this is only true since we are given a > 1, which means that $\infty^{1-a} = 0$

15. **B**: This is the construction of a full adder. First, the lower bit, S, is computed by outputting 0 if there were an even number of 1s between X, Y, and Z. This is done by XOR'ing the three together, as was done in question 5. The higher bit is set to 1 if at least 2 of the 3 of X,Y, and Z were 1, since in binary $1_2+1_2=10_2$. We can represent this by choosing each pair of X, Y, and Z and AND'ing them together. Then we OR the result of these pairwise ANDs together since only 1 of them must be true in order for us to need the higher bit, C. C = XY + XZ + YZ.